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ANALYSES OF PARALLEL FLOW, MULTI-STREAM HEAT EXCHANGERS

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Abstract—Several analyses are presented for the modeling and design of parallel flow, multi-stream heat exchangers. The first model is the most accurate although the effects of conduction have been considered only to the extent of including surface effectivenesses. Successive models are more approximate, but generally easier to use. Analyses of multi-section heat exchangers are also given. Plate-fin heat exchangers are specifically considered. The methods presented are practical with the aid of a digital computer.

NOMENCLATURE

A, size of the reference area at a given point $[l^2]$*;
A_i, total heat transfer area of the i-th stream in a section $[l^2]$;
A_{ij}, area of the i-th stream transferring heat to the j-th stream $[l^2]$;
A_{ref}, reference area $[l^2]$;
A^j_T, total reference area in the j-th section of the heat exchanger $[l^2]$;
$[B_{il}]$, temperature transfer matrix;
c_p, specific heat at constant pressure $[Q/mT]$;
C_i, $= \pm w_i c_{pi}$, directionalized capacity rate of the i-th stream $[Q/sT]$;
$D(\,)$, determinant;
f, $= m \,.\, n$;
F_{ik}, cofactor corresponding to the i-th stream and the k-th root, defined after equation (13);
h_i, heat transfer coefficient of the i-th stream $[Q/s\,l^2T]$;
H, defined by equation (32) $[Q/s\,l^2T]$;
H_i, modified heat transfer coefficient, defined by equation (31) $[Q/s\,l^2T]$;
K_l, coefficient of equation (6);
L_{ij}, number of common walls between the i-th and j-th streams in a plate-fin heat exchanger;
m, number of streams;
M_i, number of identical channels containing the i-th stream;
n, number of sections in a multi-section heat exchanger;
N_k, constant in equation (11) pertaining to the k-th root;
p, $= d/dA_{\text{ref}}$, differential operator $[l^{-2}]$;
r_k, k-th root of equation (6) $[l^{-2}]$;
t, particular part of the temperature function, defined by equations (20) and (22) $[T]$;
T_i, temperature of the i-th stream $[T]$;
$T_{0,i}$, known temperature of the i-th stream at location A_0 $[T]$;

* In the brackets, dimensions are given as follows: l-length, m-mass, s-time, T-temperature, Q-heat $[m\,l^2/s^2]$.

T', homogeneous part of the temperature function, defined by equations (20) and (21) [T];
T_{Mi}, mean stream temperature, defined by equation (38) [T];
T_{Mw}, mean wall temperature, defined by equation (39) [T];
T_w, wall temperature [T];
u_{ij}, overall heat transfer coefficient associated with A_{ij}' [Q/s l²T];
U_{ii}, defined by equation (4) [Q/s l²T];
U_{ij}, modified overall heat transfer coefficient, defined by equation (2) [Q/s l²T];
$\pm w$, directionalized mass flow rate [m/s];
y, number of constant temperature streams.

Subscripts

Subscripts identify streams except as noted
a, end point of a section where $A_{ref} = 0$;
b, end point of a section where $A_{ref} = A_T$;
e, constant temperature stream.

Superscripts

Superscripts identify sections in a multi-section heat exchanger.

Matrices

If within the brackets there are two terms separated by a comma, the first represents the elements on the principal diagonal and the second term represents the rest of the elements.

1. INTRODUCTION

THE IMPORTANCE of multi-stream heat exchangers in certain fields, such as cryogenics, has been well established. A reliable and practical analysis applicable to the modeling and actual design of such heat exchangers, however, has been lacking. Some analyses have been performed on three-stream, parallel-flow* configurations [1–3] and more general analyses were presented by Wolf [4] and Kao [5]. The last also included the effect of conduction along the fins in plate–fin heat exchangers. The three-stream studies indicate that the extension of the usual effectiveness—NTU concepts to even three streams increases the complexities of the results by a considerable degree. Whereas in the two-fluid case one effectiveness can be expressed in terms of two variables, namely the capacity rate ratio and the NTU; in the three-fluid case two effectivenesses, or temperature ratios, exist each of which depends on six variables, namely an inlet temperature ratio, two capacity rate ratios, two thermal resistance ratios, and an NTU [2, 3]. The number of parameters increases approximately as the square of the number of streams. References [4] and [5] solved directly for temperatures in parallel-flow heat exchangers in terms of some basic parameters. Unfortunately, we found major difficulties in trying to apply these methods to an actual case. As will be seen below, some of the rather involved steps in [4] can be simplified to facilitate calculations. Consideration of conduction along the fin alone, as in [5], seems unwarranted if axial conduction along the usually much thicker walls is ignored. Since the film coefficients are evaluated experimentally anyway, the temperature distribution along the fins would have a significant effect only if it becomes very strongly asymmetric, i.e. if the temperature on one side of a channel was drastically different from that on the other side. Ordinarily these temperatures are close to each other; there is a zero temperature gradient in the fins near the channel centerline; and, consequently, there is no net conduction from the wall on one side to the other.

The purpose of this work was to develop analyses and methods of calculation which could be used in practice for the prediction of the performance and for the design of such heat exchangers. It is our conviction, however, that the large number of calculations to be performed in any real situation make the use of a digital computer essentially mandatory.

* The word "parallel" here implies both uni-directional and counter-flow configurations.

2. ANALYSES OF SINGLE-SECTION HEAT EXCHANGERS

The following analyses present different methods for determining the performance of a multi-stream, parallel-flow heat exchanger shown in Fig. 1, i.e. finding the temperature distributions, especially outlet temperatures, for a given geometry and flow configuration. The procedure is given in the order required to obtain the solution from the known parameters.

2.1 *Model no.* 1, *"exact" analysis*

Consider a heat exchanger, consisting of n streams, in which any stream can exchange heat with any other stream across a separating wall. Then an overall heat transfer coefficient, u_{ij}, can be defined for the heat transferred between the i-th and j-th streams through area A_{ij}. The energy balance for the i-th stream in a differential element of the heat exchanger, designated by an arbitrary differential area $\mathrm{d}A_{\mathrm{ref}}$ can be

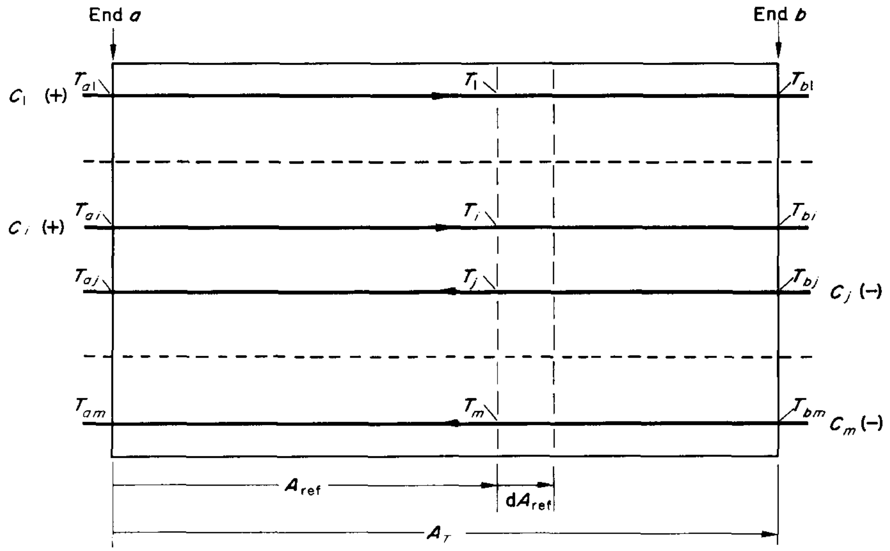

FIG. 1. Schematic diagram of a single-section heat exchanger with m streams.

The heat exchanger is assumed to operate in a steady state. Axial heat conduction is neglected. Capacity rates and overall heat transfer coefficients between pairs of streams are assumed to be constants. Variations in these parameters will be handled later by dividing the heat exchanger into sections within which the parameters can be taken as constants. Heat exchange with the environment or with any constant temperature stream will be considered later in this report.

written as

$$C_i \frac{\mathrm{d}T_i}{\mathrm{d}A_{\mathrm{ref}}} = \sum_{j=1}^{m} \frac{A_{ij}}{A_{\mathrm{ref}}} u_{ij}(T_j - T_i). \qquad (1)$$

In order for this equation to hold in all cases, the capacity rates must be directionalized, i.e. C_i is positive if the flow is in the direction of positive $\mathrm{d}A_{\mathrm{ref}}$ and C_i is negative if the flow is opposite. The positive direction is arbitrary, but we started either at the cold end of a counterflow heat exchanger, in order to obtain increasing temperatures with increasing areas, or at the end with the greater number of known inlet temperatures. The area, A_{ij}, represents only that wall area in the i-th stream which exchanges heat with the j-th stream. For example, in a plate-fin heat exchanger for the flow in a single channel, A_{ij} can be taken as half of the total channel surface area. Note that $u_{ii} = 0$.

Designate $d/dA_{\rm ref}$ by the operator p, and define new overall heat transfer coefficients, all referred to $A_{\rm ref}$, as

$$U_{ij} \equiv \frac{A_{ij}}{A_{\rm ref}} u_{ij}. \tag{2}$$

Note that $U_{ij} = U_{ji}$. Substituting into (1) and rearranging yields the general governing equation.

$$-U_{i1}T_1 - U_{i2}T_2 \ldots - U_{i,i-1}T_{i-1} + (C_i p + U_{ii})T_i - U_{i,i+1}T_{i+1} \ldots - U_{im}T_m = 0 \tag{3}$$

where

$$U_{ii} \equiv \sum_{j=1}^{m} U_{ij}. \tag{4}$$

Thus, a complete set of m simultaneous equations can be written in matrix form as

$$[C_i p + U_{ii}, -U_{ij}]\,[T_i] = 0 \tag{5}$$

where the first term in the coefficient matrix represents the diagonal elements, the second term the rest of the elements.

In order for these simultaneous equations to have a nontrivial solution, the determinant of the coefficient matrix must vanish. Setting this determinant equal to zero yields an m-th order equation in p.

$$K_m p^m + K_{m-1} p^{m-1} + \ldots + K_2 p^2 + K_1 p + K_0 = 0. \tag{6}$$

The coefficients can be evaluated as follows.

$$K_m = \prod_{i=1}^{m} C_i \tag{7}$$

$$K_l = \sum_{i_1=1}^{m-l+1} \sum_{i_2=i_1+1}^{m-l+2} \sum_{i_3=i_2+1}^{m-l+3} \sum_{i_l=i_{l-1}+1}^{m} C_{i_1} C_{i_2} C_{i_3} \ldots C_{i_l} D(U)_{i_1, i_2, i_3, \cdots i_l} \tag{8}$$

where $D(U)_{i_1, i_2, i_3, \cdots i_l}$ is the determinant of the $[U_{ij}]$ matrix with all rows and columns indicated by the subscripts eliminated. $[U_{ij}]$ contains U_{ii} terms on the principal diagonal and $-U_{ij}$ terms everywhere else. Using (7) an alternate expression can be developed, namely

$$K_l = \sum_{i_1=1}^{l+1} \sum_{i_2=i_1+1}^{l+2} \sum_{i_3=i_2+1}^{l+3} \ldots \sum_{i_{m-l}=i_{m-l-1}+1}^{m} \frac{K_m}{C_{i_1} C_{i_2} C_{i_3} \ldots C_{i_{m-l}}} D(U_{i_1, i_2, i_3, \cdots i_{m-l}}) \tag{9}$$

where $D(U_{i_1, i_2, i_3, \cdots i_{m-l}})$ is the determinant of $[U_{ij}]$ with only the rows and columns indicated by the subscript retained.

Because of equation (4) and because $u_{ii} = 0$, $K_0 = 0$, and the first root of equation (6) is $r_1 = 0$. Thus equation (6) can be rewritten as

$$K_m p^{m-1} + K_{m-1} p^{m-2} + \ldots + K_2 p + K_1 = 0 \tag{10}$$

The next step is to find the roots of this equation by any available method. Reference [4] showed that since $u_{ii} = 0$ the roots must be all simple, i.e. non-repetitive. Since the coefficient matrix in equation (5) is real and symmetric, it can be also stated that all roots are real. The rank of the coefficient matrix is $(m - 1)$, therefore, the solutions, i.e. the temperatures, are proportional to the cofactors of their coefficients in any row of the coefficient matrix (cf. [6]). The temperature of the i-th stream at some given location in the heat exchanger can be written as

$$T_i = \sum_{k=1}^{m} N_k F_{ik} \, e^{r_k A} \tag{11}$$

where the constants N_k pertain to the k-th root; F_{ik} is the cofactor, defined below, pertaining to the i-th stream and k-th root; and A is the size of the reference area at the particular location.

Since $r_1 = 0$, $F_{i1} = 1$ and (11) can be written as

$$T_i = N_1 + \sum_{k=2}^{m} N_k F_{ik} e^{r_k A}. \quad (12)$$

Substituting equation (12) into (5) and setting the coefficients of each exponential term equal to zero yields $(m - 1)$ matrix equations, one corresponding to each non-zero root, of the following form

$$[C_i r_k + U_{ii'} - U_{ij}][N_k F_{ik}] = 0. \quad (13)$$

F_{ik} is the cofactor of the coefficient in one arbitrary row and the i-th column of the coefficient matrix in (13). The constants N_k can be found by solving m equations for m boundary conditions simultaneously. In the usual case, one temperature, $T_{0,i}$, is known at some location, $A_{0,i}$, in each stream i. Thus the boundary conditions have the form

$$T_{0,i} = \sum_{k=1}^{m} N_k F_{ik} e^{r_k A_{0,i}}. \quad (14)$$

If we are interested only in end temperatures, the stream temperatures, $[T_{bi}]$, at the end of the heat exchanger where $A_{ref} = A_T$ can be expressed in terms of the stream temperatures, $[T_{ai}]$, at the other end where $A_{ref} = 0$. From equation (11) the two end temperatures can be written in matrix notation as

$$[T_{ai}] = [F_{ik}][N_k] \quad (15)$$

$$[T_{ib}] = [F_{ik} e^{r_k A_T}][N_k]. \quad (16)$$

Premultiplying both sides of equation (15) by $[F_{ik}]^{-1}$ and substituting the resulting expression for $[N_k]$ into equation (16) yields an equation for $[T_{bi}]$ in terms of $[T_{ai}]$. Since each T_{bi} depends on all of the other end temperatures T_{ai}, the subscripts for the latter have to be changed in the matrix equation.

$$[T_{bi}] = [F_{ik} e^{r_k A_T}] \; [F_{ik}]^{-1} [T_{al}]$$
$$= [B_{il}][T_{al}] \quad (17)$$

where the matrix

$$[B_{il}] \equiv [F_{ik} e^{r_k A_T}] \; [F_{ik}]^{-1} \quad (18)$$

may be called a "temperature transfer matrix" for the heat exchanger since it relates the terminal stream temperatures on either end. As will be shown later, if this temperature transfer matrix is known for each section of a multi-section heat exchanger, these may be combined in a simple fashion to determine an "overall temperature transfer matrix" for an entire multi-section heat exchanger; thus relating to each other all inlet and outlet terminal temperatures.

It should be noted that it is only necessary to know the heat transfer coefficients, heat capacity rates, and surface areas for each stream in a given heat exchanger to determine each of the elements of the matrix $[B_{il}]$. These properties are generally known or assumed at the start of a problem. Note particularily that it is not necessary to determine the constants $[N_k]$ in order to find the matrix $[B_{il}]$. Thus equation (17) represents the solution to the heat exchanger problem if we seek only the terminal temperatures.

2.2 *Model no.* 1 *with constant temperature streams*

In the modeling of most heat exchangers the many applications, such as the liquefaction of natural gas, even two-phase flows can be assigned finite capacity rates due to changes in composition and pressure along the channel. In cascade type liquefiers using refrigerants which boil or condense at constant temperatures, the above analysis must be modified to allow some of the streams to remain at constant temperature. Heat exchange between the fluids and the environment can be accounted for by introducing a pseudo-stream remaining at constant temperature throughout the entire length of the heat exchanger and having the appropriate overall heat-transfer coefficients assigned to its interaction with other streams.

Suppose that out of the total of m streams y number of streams, including stream e, remain at constant temperature in the heat exchanger section under consideration. For the rest of the $(m - y)$ streams equation (3) still holds, but the matrix equation becomes

$$[C_i p + U_{ii'} - U_{ij}][T_i] = [U_{ie}][T_e]. \quad (19)$$

The left-hand side of equation (19) contains only variable temperatures and heat-transfer coefficients between streams with variable temperatures,* whereas the right-hand side contains only constant terms. Thus, the coefficient matrix on the left-hand side is an $(m - y)$ square matrix; $[T_i]$ is an $(m - y)$ column matrix; $[U_{ie}]$ is an $(m - y) . (y)$ matrix; and $[T_e]$ is a (y) column matrix.

The solution to equation (19) can be written as

$$T_i = T'_i + t_i = \sum_{k=1}^{m-y} F_{ik} N_k \, e^{r_k A} + t_i \quad (20)$$

where the two parts of the solution satisfy the following matrix equations:

$$[C_i p + U_{ii'} - U_{ij}][T'_i] = 0 \quad (21)$$

$$[U_{ij}][t_i] = [U_{ie}][T_e]. \quad (22)$$

Since $[t_i]$ is constant, equation (22) can be solved by standard methods: e.g. Cramer's rule or by premultiplying both sides of the equation by $[U_{ij}]^{-1}$ to yield

$$[t_i] = [U_{ij}]^{-1}[U_{ie}][T_e]. \quad (23)$$

Equation (21) can be solved by the procedure outlined in the previous section. However, in this case the order of the polynomial corresponding to equation (6) is $(m - y)$, and

$$K_0 = D(U) \neq 0 \quad (24)$$

where $D(U)$ is the determinant of $[U_{ij}]$ defined after equation (8). There are $(m - y)$ unknown constants N_k which can be evaluated from boundary conditions for each variable temperature stream, as in equation (14).

Again, if we are interested only in end temperatures, the stream temperatures, $[T_{bi}]$, can be expressed in terms of $[T_{ai}]$. From equations (20) and (23)

$$[T_{ai}] = [F_{ik}][N_k] + [U_{ij}]^{-1}[U_{ie}][T_e] \quad (25)$$

$$[T_{bi}] = [F_{ik} \, e^{r_k A_T}][N_k] + [U_{ij}]^{-1}[U_{ie}][T_e]. \quad (26)$$

Premultiplying both sides of equation (25) by $[F_{ik}]^{-1}$ and substituting the resulting expression for $[N_k]$ into equation (26) yields $[T_{bi}]$ in terms of $[T_{ai}]$. Again, some of the subscripts i on the right-hand side have to be changed to l.

$$\begin{aligned}[T_{bi}] &= [F_{ik} \, e^{r_k A_T}][F_{ik}]^{-1}[T_{al}] \\ &\quad - [F_{ik} \, e^{r_k A_T}][F_{ik}]^{-1}[U_{ij}]^{-1}[U_{ie}][T_e] \\ &\quad + [U_{ij}]^{-1}[U_{ie}][T_e] \\ &= [B_{il}][T_{al}] + \{[1] - [B_{il}]\} \\ &\qquad \times [U_{ij}]^{-1}[U_{ie}][T_e]. \end{aligned} \quad (27)$$

An alternate approach is to change equation (19) by redefining $[T_i] = [T_e]$ to contain all m temperatures, including the constant ones. Then the coefficient matrices on both sides become $(m . m)$, and have to contain zeros in all rows and columns pertaining to the constant temperatures, except on the principal diagonal where the corresponding terms are unity. In addition, the coefficient matrix on the right-hand side has to be expanded by adding zeros. Thus, even though $[U_{ie}]$ is now an $(m . m)$ matrix, only the columns pertaining to the constant temperatures contain non-zero terms, namely U_{ie} or 1. With these changes $[T_e] = [T_{al}]$, and equation (27) can be written as

$$\begin{aligned}[T_{bi}] &= \{[F_{ik} \, e^{r_k A_T}][F_{ik}]^{-1} \\ &\quad - [F_{ik} \, e^{r_k A_T}][F_{ik}]^{-1}[U_{ij}]^{-1}[U_{ie}] \\ &\qquad + [U_{ij}]^{-1}[U_{ie}]\}[T_{al}]. \end{aligned} \quad (28)$$

preceding $[T_{al}]$ could be defined as the "temperature transfer matrix." This approach has the major advantage that when multi-section heat exchangers are considered, the matrices for all sections have the same size $(m . m)$.

2.3 *Model no. 2, "approximate" analysis*

In contrast to the previous sections where overall heat transfer coefficients between pairs of streams were used, here we assume that a single wall temperature, T_w, exists which varies only with axial position in the heat exchanger

* U_{ii} still contains all m heat transfer coefficients.

and which can be used with the stream temperatures, T_i, and heat transfer coefficients, h_i, to calculate the energy exchange between the wall and each stream. For steady state, the energy balance for the i-th stream in a differential element of the heat exchanger, again designated by dA_{ref}, can be written as

$$C_i \frac{dT_i}{dA_{ref}} = \frac{A_i}{A_{ref}} h_i(T_w - T_i) \qquad (29)$$

where A_i is the total heat transfer area in the i-th stream. The energy balance for the wall can be expressed as

$$\sum_{i=1}^{m} \frac{A_i}{A_{ref}} h_i(T_w - T_i) = 0. \qquad (30)$$

Define

$$H_i \equiv \frac{A_i}{A_{ref}} h_i \qquad (31)$$

$$H \equiv \sum_{i=1}^{m} H_i. \qquad (32)$$

Then equation (30) can be solved for T_w.

$$T_w = \frac{1}{H} \sum_{i=1}^{m} H_i T_i. \qquad (33)$$

Substituting equation (33) into (29), rearranging and using again the p operator yields the governing differential equation.

$$\left(\frac{H}{H_i} C_i p + H\right) T_i - \sum_{j=1}^{m} H_j T_j = 0. \qquad (34)$$

The complete set of equations in matrix form is

$$\left[\frac{H}{H_i} C_i p + H - H_i, - H_j\right] [T_i] = 0. \qquad (35)$$

In order for this simultaneous set of equations to have a non-trivial solution, the determinant of the coefficient matrix must vanish. To simplify the calculations, each column of the coefficient matrix can be divided by the corresponding $(-H_i)$ to yield a matrix

$$\left[-\frac{HC_i}{H_i^2} p + 1 - \frac{H}{H_i}, 1\right].$$

The corresponding determinant yields an m-th order polynomial in p, exactly as equation (6), except that the coefficients can be evaluated as follows.

$$K_m = (-1)^m H^m \prod_{i=1}^{m} \frac{C_i}{H_i^2} \qquad (36)$$

$$K_l = (-1)^l H^l \sum_{i_1=1}^{m-l+1} \sum_{i_2=i_1+1}^{m-l+2} \cdots \sum_{i_l=i_{l-1}+1}^{m} \frac{C_{i_1}}{H_{i_1}^2} \frac{C_{i_2}}{H_{i_2}^2} \cdots \frac{C_{i_l}}{H_{i_l}^2} D(H)_{i_1, i_2, \cdots i_l} \qquad (37)$$

where $D(H)_{i_1, i_2, \cdots i_l}$ is the determinant of the $[1 - (H/H_i), 1]$ matrix with all the rows and columns indicated by the subscripts eliminated. The first term in this matrix again represents the diagonal elements, and the second term indicates that all elements not on the diagonal are 1.

Since the rank of this coefficient matrix is also $(m - 1)$, equation (11) is true in this case too. As before with equation (13), the cofactors F_{ik} are obtained from the coefficient matrix in equations (35) by replacing p by a root r_k and using one arbitrary row and the i-th column. To find the constants N_k, again a set of m simultaneous equations of the form of equation (14) have to be solved.

One may also develop a "temperature transfer matrix" to relate to each other the temperatures at the two ends of the heat exchanger in a completely analogous fashion to that described in Section 2.1 of this paper.

2.4 *Model no. 2 with constant temperature streams*

Comparison of Models No. 1 and 2 indicates that the procedures for the two models are identical, only the terms in the coefficient matrices of the governing equations, (5) and (35), differ. Thus the discussion in Section 2.2 on Model No. 1 with constant temperature streams is applicable in this case as well if the appropriate terms are replaced as follows

Model No. 1	Model No. 2
C_i	$(H/H_i)\,C_i$
U_{ii}	$H - H_i$
U_{ij}	H_j
U_{ie}	H_e
$D(U)$	$D(H)$

2.5 *Further approximations*

To accommodate variations in fluid properties and heat transfer coefficients within the heat exchanger, it can be divided into small enough sections to make the assumption of constant parameters valid. If the heat exchanger is divided into a large number of sections, a less rigorous method may be used to determine the end temperatures of individual sections. The method is simpler than the previous ones in that no differential equations are involved, and it can be considered a finite difference approximation of the previous, more exact models.

The basic approach in such a model is to assume that the energy exchange in the entire section of the heat exchanger can be expressed in terms of a single mean fluid temperature, T_{Mi}, and a single mean wall temperature, T_{Mw}. Define these temperatures as

$$T_{Mi} \equiv \frac{T_{ai} + T_{bi}}{2} \tag{38}$$

$$T_{Mw} \equiv \frac{T_{wa} + T_{wb}}{2}. \tag{39}$$

First, using the concept of the overall heat transfer coefficient, as in Model No. 1, the energy balance for the i-th stream in a small but finite heat exchanger section can be approximated by

$$C_i(T_{bi} - T_{ai}) = \sum_{j=1}^{m} A_{ij}u_{ij}(T_{Mj} - T_{Mi}). \tag{40}$$

Dividing by $A_{\rm ref}$ and using definitions (2) and (38) yields the governing algebraic equation

$$\frac{2C_i}{A_{\rm ref}} T_{bi} + U_{ii}T_{bi} - \sum_{j=1}^{m} U_{ij}T_{bj} = \frac{2C_i}{A_{\rm ref}} T_{ai} - U_{ii}T_{ai} + \sum_{j=1}^{m} U_{ij}T_{aj}. \tag{41}$$

The complete set of m equations is

$$\left[\frac{2C_i}{A_{\rm ref}} + U_{ii}, -U_{ij}\right][T_{bi}] = \left[\frac{2C_i}{A_{\rm ref}} - U_{ii}, U_{ij}\right][T_{al}]. \tag{42}$$

Premultiplying both sides by the inverse of the left-hand coefficient matrix yields

$$[T_{bi}] = \left[\frac{2C_i}{A_{\rm ref}} + U_{ii}, U_{ij}\right]^{-1} \times \left[\frac{2C_i}{A_{\rm ref}} - U_{ii}, U_{ij}\right][T_{al}] = [B_{il}][T_{al}] \tag{43}$$

where the "temperature transfer matrix", $[B_{il}]$, is, of course, different than the previous ones.

A similar approximation can be made for Model No. 2 utilizing the mean temperatures defined by equations (38) and (39). Using also equation (33), the energy balance for the i-th stream can be written as

$$C_i(T_{bi} - T_{ai}) = \frac{H_iA_{\rm ref}}{2} \times \left[\frac{1}{H}\sum_{j=1}^{m}(T_{aj} + T_{bj})H_j - (T_{ai} + T_{bi})\right]. \tag{44}$$

This may be rewritten as

$$\left(\frac{2HC_i}{H_iA_{\rm ref}} + H\right)T_{bi} - \sum_{j=1}^{m} H_jT_{bj} = \left(\frac{2HC_i}{H_iA_{\rm ref}} - H\right)T_{ai} + \sum_{j=1}^{m} H_jT_{aj}. \tag{45}$$

The complete set of m equations is

$$\left[\frac{2HC_i}{H_iA_{\rm ref}} + H - H_i, -H_j\right][T_{bi}] = \left[\frac{2HC_i}{H_iA_{\rm ref}} - H + H_i, H_j\right][T_{al}] \tag{46}$$

from which

$$[T_{bi}] = \left[\frac{2HC_i}{H_i A_{\text{ref}}} + H - H_i, -H_j\right]^{-1} \times \left[\frac{2HC_i}{H_i A_{\text{ref}}} - H + H_i, H_j\right][T_{al}] = [B_{il}][T_{al}]. \quad (47)$$

To include constant temperature streams in both of these models, the coefficient matrices in equations (42) and (46) have to be modified. In each row and column corresponding to a constant temperature stream the matrix elements are zero except on the principal diagonal where the elements are 1. With these modifications the "temperature transfer matrices" can be evaluated as shown in equations (43) and (47).

3. ANALYSES OF MULTI-SECTION HEAT EXCHANGERS

In many applications, particularly in cryogenics, the properties of the fluids cannot be considered constant over the entire temperature range occurring within the heat exchanger. In order to apply the analyses developed previously in this paper to such a situation, the heat exchanger must be divided into several sections, such that in each section all capacity rates and heat transfer coefficients may be considered constant. Then the analyses developed previously hold for each section separately.

3.1 *Use of the "temperature transfer matrices"*

If we are interested only in inlet and outlet temperatures, the "temperature transfer matrices" previously developed may be combined in such a fashion that one can relate for the entire heat exchanger the terminal fluid temperatures to each other through a set of m simultaneous linear equations. This set of equations allows one to solve for the unknown terminal temperatures in terms of those specified for a given problem.

Let us consider a heat exchanger consisting of m streams and n sections as shown in Fig. 2. The stream temperatures at the locations of adjoining sections are identified by a superscript designating the section, and by subscripts indicating ends a or b and the stream.

If we let $[B^j_{il}]$ designate the "temperature transfer matrix" for section j as determined by any one of the procedures developed previously, then we have the following set of simultaneous equations relating the stream temperatures

$$[T^j_{bi}] = [B^j_{il}][T^j_{al}]. \quad (48)$$

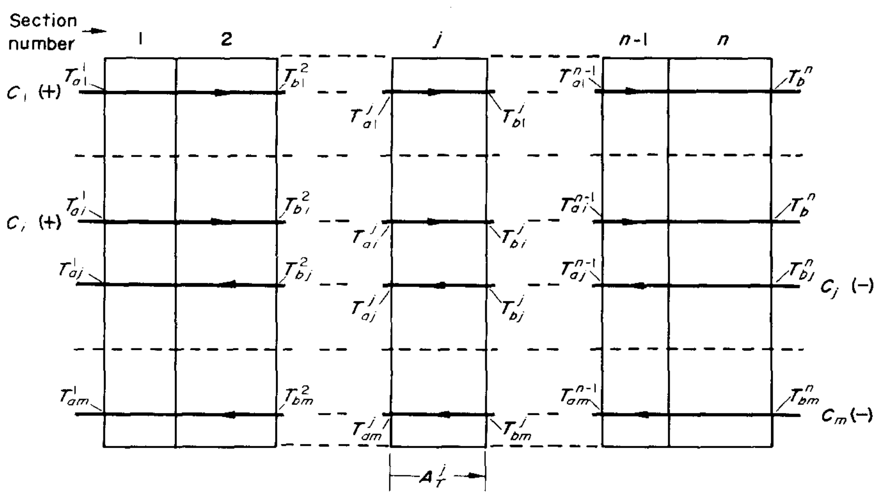

FIG. 2. Schematic diagram of a multi-section heat exchanger with m streams and n sections.

Applying (48) successively, first to section n, then section $n - 1$, etc., we may develop a set of m simultaneous equations relating the temperatures T^n_{bi} to the temperatures T^1_{ai}.

$$[T^n_{bi}] = [B^t_{il}][T^1_{al}] \quad (49)$$

where the "overall temperature transfer matrix", $[B^t_{il}]$, is a product of the "temperature transfer matrices" of all sections starting with the n-th one

$$[B^t_{il}] = [B^n_{iq}][B^{n-1}_{qr}]\ldots[B^1_{sl}]. \quad (50)$$

In solving a particular problem, the procedure is to first specify the heat transfer coefficients, capacity rates, and heat transfer surface areas for each heat exchanger section and stream. Secondly, the temperature transfer matrices $[B^j_{il}]$ would be calculated for each section based upon the specified data and utilizing the particular procedure selected for the treatment of the individual heat exchanger sections. It should be noted here that if one utilizes a less accurate method of handling a particular heat exchanger section, then one must divide the heat exchanger into more sections in order not to lose accuracy in the overall heat exchanger analysis. Thirdly, the "temperature transfer matrices" of the individual sections are combined utilizing (50) to determine an "overall temperature transfer matrix" $[B^t_{il}]$ relating the terminal fluid temperatures. Utilizing now the m specified terminal fluid temperatures at either end of the heat exchanger, one may calculate the remaining m unknown terminal fluid temperatures. It is then possible to work from either end of the heat exchanger utilizing equation (48) for each section to determine all of the stream temperatures at interior locations between heat exchanger sections.

If the heat transfer coefficients and capacity rates of each section are strongly temperature dependent and are not known precisely at the start of a problem, values may be assumed and an iterative procedure may be adopted whereby the mean fluid temperature calculated from a set of assumed fluid properties may be used to determine a new set of fluid properties for the next iteration. The criteria for convergence can be based upon a comparison of calculated mean stream temperatures from successive iterations for each stream and each section. Further iterations may be terminated when the calculated changes are less than some predetermined limit.

3.2 *Evaluation of the constants* N_k

A complete solution can be obtained by evaluating the constants N_k. If there are m streams and n sections, there are $f = m.n$ equations with the same number of boundary conditions to satisfy. Of these boundary conditions, m are usually the temperatures specified for each stream at either end of the heat exchanger, as given by equation (14). The rest of the boundary conditions arise from the requirement that the temperatures at one end of one section be equal to the temperatures of the adjacent end of the next section.

$$\sum_{k=1}^{m} N^j_k F^j_{ik}\, e^{r^j_k A^j_T} - \sum_{k=1}^{m} N^{j+1}_k F^{j+1}_{ik} = 0. \quad (51)$$

For the purpose of solving these f simultaneous equations, they can be set up in the matrix form with q known boundary conditions at the starting end grouped together at the top, and $(m\text{–}q)$ known boundary conditions at the other end grouped at the bottom. For a heat exchanger consisting of only one section, these m equations constitute the complete set to be solved. Thus the matrix equation is

$$[F^j_{ik}, F^j_{ik}\, e^{r^j_k A^j_T}, 0][N^j_k] = [T_{0,i} \text{ and } 0]. \quad (52)$$

The coefficient matrix is $(f.f)$, and the terms indicated in the brackets represent typical terms but with no specific designation as to location within the matrix. Generally, the non-zero terms are near the principal diagonal. $[N^j_k]$ and the right-hand matrix are f column matrices.

If y number of streams have constant temperatures, then equation (20) applies at the end points where the temperatures are known. For the internal boundaries, the equality of temperatures requires

$$\sum_{k=1}^{m-y} N_k^j F_{i,k}^j \, e^{r_k^j A_T^j} - \sum_{k=1}^{m-y} N_k^{j+1} F_{i,k}^{j+1} = t_i^{j+1} - t_i^j. \quad (53)$$

Therefore, the number of equations is reduced by y in each section. Note that y can be different in each section.

If a stream has variable temperature in one section, j, which becomes constant T_s in the next, $j + 1$, then the internal boundary condition becomes

$$\sum_{k=1}^{m} N_k^j F_{i,k}^j \, e^{r_k^j A_T^j} = T_s - t_i^j. \quad (54a)$$

If, on the other hand, a constant temperature stream in one section, $j - 1$, becomes one with variable temperature in the next, j, the corresponding boundary condition is

$$\sum_{k=1}^{m} N_k^j F_{i,k}^j = T_s - t_i^j. \quad (54b)$$

Obviously, if the constants N_k^j are found by solving the equations simultaneously, the temperatures at any point in the heat exchanger can be easily calculated.

4. APPLICATION TO PLATE–FIN EXCHANGER

The usual heat exchanger problem, to which the above analyses can be most directly applied, does not start with known overall heat transfer coefficients, U_{ik}, but with a given geometry; and known inlet temperatures, flow rates, and film coefficients.

The rather well-defined geometry of plate–fin heat exchangers allows the development of relatively simple routines for the calculation of the heat transfer coefficients, which are worth while discussing.

In applying the heat exchanger analysis for Model No. 1 to a plate–fin heat exchanger, we assigned half of the total surface area in a channel to each of the two sides. Because of the limitations on the size of current digital computers and on the accuracy of solving $m \, . \, n$ similtaneous equations,* it was not practical to assume that each channel constitutes a separate stream. Instead, it was assumed that a stream will have the same temperature profile in every channel that it traverses. In a well-designed heat-exchanger such an assumption is quite reasonable. If the geometry, however, indicates that such an assumption may be improper, then the heat exchanger can be split longitudinally along quasi-adiabatic boundaries, e.g. between two channels with the same stream; and the resulting parts can be treated as independent heat exchangers.

For each section the heat transfer areas associated with each stream in conjunction with every other stream have to be calculated. For example, if stream 1 shares common walls with streams 2–4, then it has three separate areas: A_{12}, A_{13} and A_{14}. In general, if the geometries are the same for all channels containing stream i,

$$A_{ij}^c = \frac{A_i^c}{2M_i} L_{ij} \quad (55)$$

where A_i^c is the total heat transfer surface of stream i in the section c, M_i is the number of identical channels containing stream i, and L_{ij} is the number of common walls between streams i and j. However, if one side of a channel is adiabatic, or if several neighboring channels contain the same stream i with a quasi-adiabatic line along the center of such a channel-group; then the areas on each side of the adiabatic line can be assigned to the heat exchange with the next stream on the same side. A correction factor should be used, however, to account for the reduction in fin effectiveness due to increased length.

The overall heat transfer coefficients can be

* For the computer facilities we used with double precision $m \, . \, n < \sim 90$.

calculated from

$$\frac{1}{U_{ij}A_{\text{ref}}} = \frac{1}{h_i A_{ij}} + \frac{1}{h_j A_{ji}} \tag{56}$$

where the film coefficients, h_i and h_j, include the temperature effectivenesses of the respective total areas. Note that A_{ij} is not necessarily equal to A_{ji}. With the known overall heat transfer coefficients, U_{ij}, the problem can be solved as outlined before.

5. DISCUSSION

It seems obvious that the practicality of these analyses depends on the use of a well-developed program for a digital computer. We endeavored to provide enough details to allow the reader to develop his own program. We found it necessary to use double precision throughout, and we employed available or modified library routines for standard operations, such as finding the roots of equation (6).

For well-designed heat exchangers all methods yield very similar results. If for some reason a heat exchanger had very asymmetric temperatures occurring at certain cross sections, then models which are based on the concept of a common wall temperature at each cross section, should be expected to give less accurate results than models based on overall heat transfer coefficients.

It was difficult to find adequate comparisons with existing theoretical or experimental results. Our Model No. 1 yielded results identical with those calculated in [2, 3] for a three-stream heat exchanger. Since the two analyses are identical for a three-stream heat exchanger, this similarity was expected. Example 1 of [5] gave some numerical results, but the width of the heat exchanger was not given, and the numbers, as presented, do not satisfy the heat balance. Comparisons with field data obtained on heat exchangers used in liquid natural gas plants of the Chicago Bridge and Iron Company showed reasonable agreements, with the predicted and measured temperatures generally falling within 5°F of each other. Although the measured temperatures were quite accurate, the flow rates obtained in these field data were not, and thus measured data is not presented here.

6. CONCLUSIONS

The purpose of this work was to provide a tool by which specific heat exchanger configurations can be evaluated and compared. Because of the large number of variables involved, it is virtually impossible to present any meaningful correlations. In general, we found that well balanced heat exchangers (in terms of hot and cold heat capacity rates) and good mixing of hot and cold streams (with a minimum of identical streams in thermal contact) provided the best performance.

The analyses of heat exchangers described here are based on solid, well established foundations. However, additional, carefully performed experimental data are still needed since one of the biggest difficulties is usually the uncertainty of the input data, particularly the heat transfer coefficients.

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ETUDE D'ECHANGEURS DE CHALEUR A ECOULEMENT PARALLELE ET A PLUSIEURS CANAUX

Résumé—On présente plusieurs analyses concernant la conception et la construction d'échangeurs de chaleur à écoulement parallèle et à multi-canaux. Le premier modèle est le plus efficace bien que les effets de conduction n'aient été considérés que pour tenir compte des efficacités superficielles. Des modèles successifs sont plus approximatifs, mais généralement plus faciles à utiliser. On a donné également des analyses d'échangeurs de chaleur à multi-sections. Les méthodes présentées sont utilisables à l'aide d'un calculateur digital.

UNTERSUCHUNG VON GLEICHSTROM-WÄRMEAUSTAUSCHERN, DIE VON MEHREREN FLUIDEN DURCHSTRÖMT WERDEN

Zusammenfassung—Es werden mehrere analytische Ersatzmodelle für Gleichstrom-Wärmeaustauscher, die von mehreren Fluiden durchströmt werden, untersucht. Das erste Modell ist das exakteste, obwohl Wärmeleitungseffekte nur durch Einbeziehung der Rippenwirkungsgrade berücksichtigt wurden. Die weiteren Modelle stellen nur Näherungen dar, sind aber im allgemeinen leichter zu handhaben. In mehrere Abschnitte unterteilte Wärmeaustauscher werden ebenfalls analytisch untersucht.

Besonderes Interesse gilt Wärmeaustauschern mit plattenförmigen Rippen. Die vorgeführten Methoden lassen sich mit Hilfe eines Digitalrechners anwenden.

АНАЛИЗ МНОГОПОТОЧНЫХ ТЕПЛООБМЕННИКОВ С ПАРАЛЛЕЛЬНЫМ ПОТОКОМ

Аннотация—Рассмотрено несколько вариантов моделирования и расчёта многопоточных теплообменников с параллельным током. Первая модель является наиболее точной, хотя влияние теплопроводности рассматривалось только для учёта эффективности поверхности. Следующие две модели являются более приближенными, но более удобны для пользования. Рассмотрены также многосекционные теплообменники. Особое внимание уделено пластинчатоплавниковым теплообменникам. Расчёт по предложенным методикам производится на ЭВМ.